

ELKO, flagpole and flag-dipole spinor fields, and the instanton Hopf fibration

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In a previous paper we explicitly constructed a mapping that leads Dirac spinor fields to the dual-helicity eigenspinors of the charge conjugation operator (ELKO spinor fields). ELKO spinor fields are prime candidates for describing dark matter, and belong to a wider class of spinor fields, the so-called flagpole spinor fields, corresponding to the class-(5), according to Lounesto spinor field classification, based on the relations and values taken by their associated bilinear covariants. Such a mapping between Dirac and ELKO spinor fields was obtained in an attempt to extend the Standard Model in order to encompass dark matter. Now we prove that such a mapping, analogous to the instanton Hopf fibration map $S^3 \dots S^7 \rightarrow S^4$, prevents ELKO to describe the instanton, giving a suitable physical interpretation to ELKO. We review ELKO spinor fields as type-(5) spinor fields under the Lounesto spinor field classification, explicitly computing the associated bilinear covariants. This paper is also devoted to investigate some formal aspects of the flag-dipole spinor fields, which correspond to the class-(4) under the Lounesto spinor field classification. In addition, we prove that type-(4) spinor fields — corresponding to flag-dipoles — and ELKO spinor fields — corresponding to flagpoles — can also be entirely described in terms of the Majorana and Weyl spinor fields. After all, by choosing a projection endomorphism of the spacetime algebra $\mathcal{Cl}_{1,3}$ it is shown how to obtain ELKO, flagpole, Majorana and Weyl spinor fields, respectively corresponding to type-(5) and -(6) spinor fields, uniquely from limiting cases of a type-(4) — flag-dipole — spinor field, in a similar result obtained by Lounesto.

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I. INTRODUCTION

ELKO — *Eigenspinoren des Ladungskonjugationsoperators* — spinor fields¹ represent an extended set of Majorana spinor fields, describing a non-standard Wigner class of fermions, in which the charge conjugation and the parity operators commute, rather than anticommute [1, 2, 3, 4]. Although in the algebraic framework there is no essential difference between ELKO and Majorana spinor fields (in the Lounesto spinor field classification), from the physical point of view Ahluwalia and Grumiller showed in [1] that ELKO spinor fields are competing candidates for the Majorana fields. ELKO spinor fields carry mass dimension one, and not three-halves, and consequently cannot be part of the $SU(2)_L$ doublets of the Standard Model, which includes spin-1/2 particles of mass dimension three-halves. Besides, a quantum field theory constructed for ELKO spinor fields gives a non-local character to ELKO. Ahluwalia and Grumiller argued that what localizes otherwise extended field configurations like solitons is a conserved topological charge, and in the absence of it, there is nothing that protects the particle from spreading [1]. Non-locality is related to a classical field (soliton) configuration, but the non-locality which the current papers about ELKO refer, appears at the level of the field anticommutators. Concerning the fundamental anticommutators for the ELKO quantum field, such a non-locality is at the second order, in the sense that while the field-momentum anticommutator exhibits the usual form expected of a local quantum field theory, the field-field and momentum-momentum anticommutators do not vanish [1]. In addition, the vacuum expectation value is computed to be non-trivial [1]. For more details, see also [2, 3, 4].

Further, ELKO accomplishes dual-helicity eigenspinors of the spin-1/2 charge conjugation operator, and carry mass dimension one, besides having non-local properties [1, 2, 3, 4].

It is well known that all *spinors* in Minkowski spacetime can be given — from the classical viewpoint² — as elements of the carrier spaces of the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $SL(2, \mathbb{C})$. However, according to [1, 2] only in the low-energy limit, *ELKO spinor fields* carries a representation of the Lorentz group. P. Lounesto, in the classification of spinor fields, proved that any spinor field belongs to one of the six (disjoint) classes found by him [5, 6]. Such an *algebraic* classification is based on the values assumed by their bilinear covariants, the Fierz identities, aggregates and boomerangs [5, 6, 7]. Lounesto spinor field classification has wide applications in cosmology and astrophysics (via ELKO, for instance see [1, 2, 7, 8, 9]), and in General Relativity: it was recently demonstrated that Einstein-Hilbert, the Einstein-Palatini, and the Holst³ actions can be derived from the quadratic spinor lagrangian (that describes supergravity) [11, 12], when the three classes of Dirac spinor fields, under Lounesto spinor field classification, are considered [13, 14, 15, 16]. It was also shown [7] that ELKO represents the class-(5), which also incorporates Majorana spinor fields, and that those spinor fields covers one of the six disjoint classes in Lounesto spinor field classification. Although in [7] it was not found any *algebraic* difference between ELKO and Majorana spinor fields, physically ELKO describes spin-1/2 fields presenting mass dimension one.

Any invertible map that takes Dirac spinor fields and leads to ELKO is also capable to make mass dimension transmutations, since Dirac spinor fields present mass dimension three-halves, instead of the mass dimension one associated with ELKO. The main physical motivation of a previous paper [17]⁴ was to provide the initial pre-requisites to construct a natural extension of the Standard Model (SM) in order to incorporate ELKO, and consequently a possible description of dark matter [1, 2, 8] in this context. The explicit application describing the mapping between ELKO and Dirac spinor fields obtained in [17] presents an analogy to the instanton Hopf fibration map $S^3 \dots S^7 \rightarrow S^4$ mapping obtained in [18, 19, 20], and could be interpreted as the geometric meaning of the mass dimension-transmuting operator obtained in [17]. It would suggest the reason why the ELKO spinor fields satisfy the Klein-Gordon equation, instead of the Dirac equation, as detailed and extensively shown in [1]. Physically, as ELKO presents mass dimension one [1, 2, 8], while any other type of spin-1/2 spinor field presents mass dimension 3/2, the conditions obtained in Sec. V might introduce the geometric explanation for this physical open problem.

In this paper, we are also interested in geometric and algebraic properties of flag-dipole (type-(4)) spinor fields. It is shown that such a spinor field can be written as a “sum” of Weyl and Majorana spinor fields⁵. In fact, such systematization concerning type-(4) spinor fields can be relevant in physics, since it can be related to the quark confinement problem [5]. It is also shown how to obtain type-(5) and -(6) spinor fields as limiting cases of type-(4) spinor fields. Such results are based on some previous results already obtained by Lounesto [5, 6], here used also to extend it to the ELKO algebraic and geometric formalism.

¹ ELKO is the German acronym for Dual-helicity eigenspinors of the charge conjugation operator [1].

² It is well known that spinors have three different, although equivalent, definitions: the operator, the classical and the algebraic one [22, 23, 28, 38, 39, 41, 42, 43, 44, 45].

³ The Holst action is shown to be equivalent to the Ashtekar formulation of Quantum Gravity [10].

⁴ R. da Rocha thanks to Prof. Dharamvir Ahluwalia-Khalilova for private communication on the subject.

⁵ Remember that the class of the spinor, in the Lounesto’s sense, is not necessarily preserved upon sum.

The paper is organized as follows: after briefly presenting some essential algebraic and geometric preliminaries in Section II, we introduce in Section III the bilinear covariants together with the Fierz identities. Also, the Lounesto classification of spinor fields is presented together with the definition of ELKO spinor fields [1], showing that ELKO is indeed a flagpole spinor field with opposite (dual) helicities [1, 2, 7]. We carefully show the computations leading to the classification of ELKO spinor fields as flagpole spinor fields, in the class-(5) under the Lounesto spinor field classification, for the first time proved in [7]. In Section IV the spacetime algebra and the construction of ideal and operator spinor fields is reviewed, in order to introduce the investigation of the geometric and algebraic aspects related to the flag-dipole spinor fields in Section VI. In Section V the instanton Hopf fibration $S^3 \dots S^7 \rightarrow S^4$ is discussed in the light of the bilinear covariants related to Dirac spinor fields, and the relationship between such a map and the mapping that leads Dirac spinor fields to ELKO spinor fields is considered to show that the instanton cannot be described by an ELKO. Some physical consequences are also discussed in this context. In the Appendix A the definition of operator spinors is recapitulated.

II. PRELIMINARIES

Let V be a finite n -dimensional real vector space and V^* denotes its dual. We consider the tensor algebra $\oplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \oplus_{k=0}^n \Lambda^k(V)$ of *multivectors*⁶ over V . $\Lambda^k(V)$ denotes the space of the antisymmetric k -tensors. Given $\psi \in \Lambda(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi} = (-1)^{[k/2]} \psi$ ($[k]$ denotes the integer part of k). If V is endowed with a non-degenerate, symmetric, bilinear map $\mathbf{g} : V \times V \rightarrow \mathbb{R}$, it is possible to extend \mathbf{g} to $\Lambda(V)$. Given $\psi = \mathbf{u}^1 \wedge \dots \wedge \mathbf{u}^k$ and $\phi = \mathbf{v}^1 \wedge \dots \wedge \mathbf{v}^l$, for $\mathbf{u}^i, \mathbf{v}^j \in V$, one defines $\mathbf{g}(\psi, \phi) = \det(\mathbf{g}(\mathbf{u}^i, \mathbf{v}^j))$ if $k = l$ and $\mathbf{g}(\psi, \phi) = 0$ if $k \neq l$. The projection of a multivector $\psi = \psi_0 + \psi_1 + \dots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its p -vector part is given by $\langle \psi \rangle_p = \psi_p$. Given $\psi, \phi, \xi \in \Lambda(V)$, the *left contraction* is defined implicitly by $\mathbf{g}(\psi \lrcorner \phi, \xi) = \mathbf{g}(\phi, \tilde{\psi} \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $\mathbf{v} \lrcorner a = 0$. The *right contraction* is analogously defined by $\mathbf{g}(\psi \lrcorner \phi, \xi) = \mathbf{g}(\phi, \psi \wedge \tilde{\xi})$. Both contractions are related by $\mathbf{v} \lrcorner \psi = -\tilde{\psi} \lrcorner \mathbf{v}$. The Clifford product between $\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \lrcorner \psi$. The Grassmann algebra $(\Lambda(V), \mathbf{g})$ endowed with the Clifford product is denoted by $\mathcal{Cl}(V, \mathbf{g})$ or $\mathcal{Cl}_{p,q}$, the Clifford algebra⁷ associated with $V \simeq \mathbb{R}^{p,q}$, $p + q = n$. In what follows \mathbb{R}, \mathbb{C} denote respectively the real and complex numbers.

Now, restricting to the case where $(p, q) = (1, 3)$ we briefly recall the geometry of Clifford and spin-Clifford bundles. For more details, see e.g. [23]. we denote by $\mathcal{M} = (M, \mathbf{g}, \nabla, \tau\mathbf{g}, \uparrow)$ the spacetime structure: M denotes a 4-dimensional manifold, $\mathbf{g} \in \sec T_2^0 M$ is the metric and in what follows we denote by $g \in \sec T_2^0 M$ the corresponding metric of the cotangent bundle⁸, ∇ is the Levi-Civita connection of \mathbf{g} , $\tau\mathbf{g} \in \sec \Lambda^4(T^*M)$ defines a spacetime orientation and \uparrow refers to an equivalence class of timelike 1-form fields defining a time orientation. By $F(M)$ we mean the (principal) bundle of frames, by $\mathbf{P}_{\text{SO}_{1,3}^e}(M)$ the orthonormal frame bundle, and $P_{\text{SO}_{1,3}^e}(M)$ denotes the orthonormal coframe bundle. We consider M a spin manifold, and then there exists $\mathbf{P}_{\text{Spin}_{1,3}^e}(M)$ and $P_{\text{Spin}_{1,3}^e}(M)$ which are respectively the spin frame and the spin coframe bundles. We denote by $s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow P_{\text{SO}_{1,3}^e}(M)$ the fundamental mapping present in the definition of $P_{\text{Spin}_{1,3}^e}(M)$. A spin structure on M consists of a principal fiber bundle $\pi_s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow M$, with group $\text{Spin}_{1,3}^e$, and the map

$$s : P_{\text{Spin}_{1,3}^e}(M) \rightarrow P_{\text{SO}_{1,3}^e}(M)$$

satisfying the following conditions:

- (i) $\pi(s(p)) = \pi_s(p)$, $\forall p \in P_{\text{Spin}_{1,3}^e}(M)$; π is the projection map of the bundle $P_{\text{SO}_{1,3}^e}(M)$.
- (ii) $s(p\phi) = s(p)\text{Ad}_\phi$, $\forall p \in P_{\text{Spin}_{1,3}^e}(M)$ and $\text{Ad} : \text{Spin}_{1,3}^e \rightarrow \text{Aut}(\mathcal{Cl}_{1,3})$, $\text{Ad}_\phi : \mathcal{Cl}_{1,3} \ni \Xi \mapsto \phi \Xi \phi^{-1} \in \mathcal{Cl}_{1,3}$ [23].

We recall now that sections of $P_{\text{SO}_{1,3}^e}(M)$ are orthonormal coframes, and that sections of $P_{\text{Spin}_{1,3}^e}(M)$ are also orthonormal coframes such that two coframes differing by a 2π rotation are distinct and two coframes differing by a 4π rotation are identified. Next we introduce the Clifford bundle of differential forms $\mathcal{Cl}(M, g)$, which is a vector bundle associated with $P_{\text{Spin}_{1,3}^e}(M)$. Their sections are sums of non-homogeneous differential forms, which will be called Clifford fields. We recall that $\mathcal{Cl}(M, g) = P_{\text{SO}_{1,3}^e}(M) \times_{\text{Ad}'} \mathcal{Cl}_{1,3}$, where $\mathcal{Cl}_{1,3} \simeq \text{M}(2, \mathbb{H})$ is the spacetime algebra [24]. Details of the bundle structure are as follows [23, 25]:

⁶ $\Lambda(V^*) = \oplus_{k=0}^n \Lambda^k(V^*)$ denotes the space of the antisymmetric k -cotensors, isomorphic to the k -forms vector space.

⁷ If $\mathbf{g} : V^* \times V^* \rightarrow \mathbb{R}$ we can analogously also construct the Clifford algebra $\mathcal{Cl}(V^*, \mathbf{g})$, which is of multicovectors, which plays a significant role when we consider the algebra bundle of multiform fields

⁸ If in an arbitrary basis $\mathbf{g} = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$ and $g = g^{\alpha\beta} \partial_\alpha \otimes \partial_\beta$ then $g^{\alpha\beta} g_{\beta\gamma} = \delta_\gamma^\alpha$.

(1) Let $\pi_c : \mathcal{C}\ell(M, g) \rightarrow M$ be the canonical projection of $\mathcal{C}\ell(M, g)$ and let $\{U_\alpha\}$ be an open covering of M . There are trivialization mappings $\psi_i : \pi_c^{-1}(U_i) \rightarrow U_i \times \mathcal{C}\ell_{1,3}$ of the form $\psi_i(p) = (\pi_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p))$. If $x \in U_i \cap U_j$ and $p \in \pi_c^{-1}(x)$, then

$$\psi_{i,x}(p) = h_{ij}(x) \psi_{j,x}(p) \quad (1)$$

for $h_{ij}(x) \in \text{Aut}(\mathcal{C}\ell_{1,3})$, where $h_{ij} : U_i \cap U_j \rightarrow \text{Aut}(\mathcal{C}\ell_{1,3})$ are the transition mappings of $\mathcal{C}\ell(M, g)$. We recall that every automorphism of $\mathcal{C}\ell_{1,3}$ is *inner*. Then,

$$h_{ij}(x) \psi_{j,x}(p) = a_{ij}(x) \psi_{i,x}(p) a_{ij}(x)^{-1}$$

for some $a_{ij}(x) \in \mathcal{C}\ell_{1,3}^*$, the group of invertible elements of $\mathcal{C}\ell_{1,3}$.

(2) As it is well known, the group $\text{SO}_{1,3}^e$ has a natural extension in the Clifford algebra $\mathcal{C}\ell_{1,3}$. Indeed, we know that $\mathcal{C}\ell_{1,3}^*$ (the group of invertible elements of $\mathcal{C}\ell_{1,3}$) acts naturally on $\mathcal{C}\ell_{1,3}$ as an algebra automorphism through its adjoint representation. A set of *lifts* of the transition functions of $\mathcal{C}\ell(M, g)$ is a set of elements $\{a_{ij}\} \subset \mathcal{C}\ell_{1,3}^*$ such that, if ⁹

$$\begin{aligned} \text{Ad} : \phi &\mapsto \text{Ad}_\phi \\ \text{Ad}_\phi(\Xi) &= \phi \Xi \phi^{-1}, \quad \forall \Xi \in \mathcal{C}\ell_{1,3}, \end{aligned}$$

then $\text{Ad}_{a_{ij}} = h_{ij}$ in all intersections.

(3) Also $\sigma = \text{Ad}|_{\text{Spin}_{1,3}^e}$ defines a group homeomorphism $\sigma : \text{Spin}_{1,3}^e \rightarrow \text{SO}_{1,3}^e$ which is onto with kernel \mathbb{Z}_2 . We have that $\text{Ad}_{-1} = \text{identity}$, and so $\text{Ad} : \text{Spin}_{1,3}^e \rightarrow \text{Aut}(\mathcal{C}\ell_{1,3})$ descends to a representation of $\text{SO}_{1,3}^e$. Let us call Ad' this representation, i.e., $\text{Ad}' : \text{SO}_{1,3}^e \rightarrow \text{Aut}(\mathcal{C}\ell_{1,3})$. Then we can write $\text{Ad}'_{\sigma(\phi)} \Xi = \text{Ad}_\phi \Xi = \phi \Xi \phi^{-1}$.

(4) It is clear that the structure group of the Clifford bundle $\mathcal{C}\ell(M, g)$ is reducible from $\text{Aut}(\mathcal{C}\ell_{1,3})$ to $\text{SO}_{1,3}^e$. The transition maps of the principal bundle of oriented Lorentz cotetrads $P_{\text{SO}_{1,3}^e}(M)$ can thus be (through Ad') taken as transition maps for the Clifford bundle. We then have [26]

$$\mathcal{C}\ell(M, g) = P_{\text{SO}_{1,3}^e}(M) \times_{\text{Ad}'} \mathcal{C}\ell_{1,3},$$

i.e., the Clifford bundle is a vector bundle associated with the principal bundle $P_{\text{SO}_{1,3}^e}(M)$ of orthonormal Lorentz coframes.

Recall that $\mathcal{C}\ell(T_x^*M, g_x)$ is also a vector space over \mathbb{R} which is isomorphic to the exterior algebra $\Lambda(T_x^*M)$ of the cotangent space and $\Lambda(T_x^*M) = \bigoplus_{k=0}^4 \Lambda^k(T_x^*M)$, where $\Lambda^k(T_x^*M)$ is the $\binom{4}{k}$ -dimensional space of k -forms over a point x on M . There is a natural embedding $\Lambda(T^*M) \hookrightarrow \mathcal{C}\ell(M, g)$ [26] and sections of $\mathcal{C}\ell(M, g)$ — Clifford fields — can be represented as a sum of non-homogeneous differential forms. Let $\{e_a\} \in \text{sec } \mathbf{P}_{\text{SO}_{1,3}^e}(M)$ (the orthonormal frame bundle) be a tetrad basis for $TU \subset TM$ (given an open set $U \subset M$). Moreover, let $\{\vartheta^a\} \in \text{sec } P_{\text{SO}_{1,3}^e}(M)$. Then, for each $a = 0, 1, 2, 3$, $\vartheta^a \in \text{sec } \Lambda^1(T^*M) \hookrightarrow \text{sec } \mathcal{C}\ell(M, g)$. We recall next the crucial result [23] that in a spin manifold we have:

$$\mathcal{C}\ell(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathcal{C}\ell_{1,3}.$$

Spinor fields are sections of vector bundles associated with the principal bundle of spinor coframes. The well known Dirac spinor fields are sections of the bundle

$$S_c(M, g) = P_{\text{Spin}_{1,3}^e}(M) \times_{\mu_c} \mathbb{C}^4,$$

with μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{Spin}_{1,3}^e \cong \text{SL}(2, \mathbb{C})$ in $\text{End}(\mathbb{C}^4)$ [27].

III. BILINEAR COVARIANTS AND ELKO SPINOR FIELDS

This Section is devoted to recall the bilinear covariants, using the programme introduced in [7], which we briefly recall here. In this article all spinor fields live in Minkowski spacetime $(M, \eta, D, \tau_\eta, \uparrow)$. The manifold $M \simeq \mathbb{R}^4$, η denotes a

⁹ Recall that $\text{Spin}_{1,3}^e = \{\phi \in \mathcal{C}\ell_{1,3}^0 : \phi \tilde{\phi} = 1\} \simeq \text{SL}(2, \mathbb{C})$ is the universal covering group of the restricted Lorentz group $\text{SO}_{1,3}^e$. Notice that $\mathcal{C}\ell_{1,3}^0 \simeq \mathcal{C}\ell_{3,0} \simeq \text{M}(2, \mathbb{C})$, the even subalgebra of $\mathcal{C}\ell_{1,3}$ is the Pauli algebra.

constant metric, where $\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, D denotes the Levi-Civita connection associated with η , M is oriented by the 4-volume element τ_η and time-oriented by \uparrow . Here $\{x^\mu\}$ denotes global coordinates in the Einstein-Poincaré gauge, naturally adapted to an inertial reference frame $\mathbf{e}_0 = \partial/\partial x^0$. Let $\mathbf{e}_i = \partial/\partial x^i$, $i = 1, 2, 3$. Also, $\{\mathbf{e}_\mu\}$ is a section of the frame bundle $\mathbf{P}_{\text{SO}_{1,3}}(M)$ and $\{\mathbf{e}^\mu\}$ is its reciprocal frame satisfying $\eta(\mathbf{e}^\mu, \mathbf{e}_\nu) := \mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$. Let moreover be $\{\theta^\mu\}$ and $\{\theta_\mu\}$ be respectively the dual bases of $\{\mathbf{e}_\mu\}$ and $\{\mathbf{e}^\mu\}$. Classical spinor fields carrying a $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^e$ are sections of the vector bundle $\mathbf{P}_{\text{Spin}_{1,3}^e}(M) \times_\rho \mathbb{C}^4$, where ρ stands for the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^e$ in \mathbb{C}^4 . In addition, classical spinor fields carrying a $D^{(1/2,0)}$ or $D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^e$ are sections of the vector bundle $\mathbf{P}_{\text{Spin}_{1,3}^e}(M) \times_{\rho'} \mathbb{C}^2$, where ρ' stands for the $D^{(1/2,0)}$ or the $D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^e$ in \mathbb{C}^2 . Given a spinor field $\psi \in \text{sec } \mathbf{P}_{\text{Spin}_{1,3}^e}(M) \times_\rho \mathbb{C}^4$ the bilinear covariants may be taken as the following sections of the exterior algebra bundle of *multiform* fields $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \eta)$ (where $\mathcal{C}\ell(M, \eta)$ is the Clifford bundle of multiform fields and η , of course, denotes the metric of the cotangent bundle [23]):

$$\begin{aligned} \sigma &= \psi^\dagger \gamma_0 \psi, \quad \mathbf{J} = J_\mu \theta^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \theta^\mu, \quad \mathbf{S} = S_{\mu\nu} \theta^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \theta^\mu \wedge \theta^\nu, \\ \mathbf{K} &= K_\mu \theta^\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi \theta^\mu, \quad \omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi, \end{aligned} \quad (2)$$

The set $\{\gamma_\mu\}$ refers to the Dirac matrices in chiral representation (see Eq.(5)). Also $\{\mathbf{1}_4, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\rho, \gamma_0 \gamma_1 \gamma_2 \gamma_3\}$ ($\mu, \nu, \rho = 0, 1, 2, 3$, and $\mu < \nu < \rho$) is a basis for $\mathbb{C}(4)$ satisfying [5] $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbf{1}_4$ and the Clifford product is denoted by juxtaposition. More details on notations can be found in [23, 28].

In the case of the electron, described by Dirac spinor fields (classes 1, 2 and 3 below), \mathbf{J} is a future-oriented timelike current vector which gives the current of probability, the bivector \mathbf{S} is associated with the distribution of intrinsic angular momentum, and the spacelike vector \mathbf{K} is associated with the direction of the electron spin. For a detailed discussion concerning such entities, their relationships and physical interpretation, and generalizations, see, e.g., [5, 6, 29, 31, 32, 33, 34].

The bilinear covariants satisfy the Fierz identities [5, 6, 32, 33, 34]

$$\mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{K}^2 = -\mathbf{J}^2, \quad \mathbf{J} \cdot \mathbf{K} = 0, \quad \mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma \gamma_{0123}) \mathbf{S}. \quad (3)$$

A spinor field such that *not both* ω and σ are null is said to be regular. When $\omega = 0 = \sigma$, a spinor field is said to be *singular*.

Lounesto spinor field classification is given by the following spinor field classes [5, 6], where in the first three classes it is implicit that $\mathbf{J}, \mathbf{K}, \mathbf{S} \neq 0$:

- 1) $\sigma \neq 0, \quad \omega \neq 0$.
- 2) $\sigma \neq 0, \quad \omega = 0$.
- 3) $\sigma = 0, \quad \omega \neq 0$.
- 4) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0$.
- 5) $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0$.
- 6) $\sigma = 0 = \omega, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0$.

The current density \mathbf{J} is always non-zero. Types-(1), -(2), and -(3) spinor fields are denominated *Dirac spinor fields* for spin-1/2 particles and types-(4), -(5), and -(6) are respectively called *flag-dipole*, *flagpole*¹⁰ and *Weyl spinor fields*. Majorana spinor fields are a particular case of a type-(5) spinor field. It is worthwhile to point out a peculiar feature of types-(4), -(5) and -(6) spinor fields: although \mathbf{J} is always non-zero, $\mathbf{J}^2 = -\mathbf{K}^2 = 0$. It shall be seen below that the bilinear covariants related to an ELKO spinor field, satisfy $\sigma = 0 = \omega, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0$ and $\mathbf{J}^2 = 0$. Since Lounesto proved that there are *no* other classes based on distinctions among bilinear covariants, ELKO spinor fields must belong to one of the disjoint six classes.

Types-(1), -(2) and -(3) Dirac spinor fields (DSFs) have different algebraic and geometrical characters, and we would like to emphasize the main differing points. For more details, see e.g. [5, 6]. Recall that if the quantities

¹⁰ Such spinor fields are constructed by a null 1-form field current and an also null 2-form field angular momentum, the “flag” [35].

$P = \sigma + \mathbf{J} + \gamma_{0123}\omega$ and $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$ are defined [5, 6], in type-(1) DSF we have $P = -(\omega + \sigma\gamma_{0123})^{-1}\mathbf{K}Q$ and also $\psi = -i(\omega + \sigma\gamma_{0123})^{-1}\psi$. In type-(2) DSF, P is a multiple of $\frac{1}{2\sigma}(\sigma + \mathbf{J})$ and looks like a proper energy projection operator, commuting with the spin projector operator given by $\frac{1}{2}(1 - i\gamma_{0123}\mathbf{K}/\sigma)$. Also, $P = \gamma_{0123}\mathbf{K}Q/\sigma$. Further, in type-(3) DSF, $P^2 = 0$ and $P = \mathbf{K}Q/\omega$. The introduction of the spin-Clifford bundle makes it possible to consider all the geometric and algebraic objects — the Clifford bundle, spinor fields, differential form fields, operators and Clifford fields — as being elements of an unique unified formalism. It is well known that spinor fields have three different, although equivalent, definitions: the operator, the classical and the algebraic one. In particular, the operatorial definition allows us to factor — up to sign — the DSF ψ as $\psi = (\sigma + \omega\gamma_{0123})^{-1/2}R$, where $R \in \text{Spin}_{1,3}^c(M) \hookrightarrow \mathcal{C}\ell(M, \eta)$. Finally, to a Weyl spinor field ξ (type-(6)) with bilinear covariants \mathbf{J} and \mathbf{K} , two Majorana spinor fields $\psi_{\pm} = \frac{1}{2}(\xi + C(\xi))$ can be associated, where C denotes the charge conjugation operator. Penrose flagpoles are implicitly defined by the equation $\sigma + \mathbf{J} + i\mathbf{S} - i\gamma_{0123}\mathbf{K} + \gamma_{0123}\omega = \frac{1}{2}(\mathbf{J} \mp i\mathbf{S}\gamma_{0123})$ [5, 6]. For a physically useful discussion regarding the disjoint classes -(5) and -(6) see, e.g., [36]. The fact that two Majorana spinor fields ψ_{\pm} can be written in terms of a Weyl type-(6) spinor field $\psi_{\pm} = \frac{1}{2}(\xi + C(\xi))$, is an ‘accident’ when the (Lorentzian) spacetime has $n = 4$ — the present case — or $n = 6$ dimensions. The more general assertion concerns the property that two Majorana, and more generally ELKO spinor fields ψ_{\pm} can be written in terms of a *pure spinor* field [37, 38] — hereon denoted by \mathbf{u} — as $\psi_{\pm} = \frac{1}{2}(\mathbf{u} + C(\mathbf{u}))$. It is well known that Weyl spinor fields are pure spinor fields when $n = 4$ and $n = 6$. When the complexification $\mathbb{C} \otimes \mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$ is considered, one can consider a maximal totally isotropic subspace N of $\mathbb{C} \otimes \mathbb{R}^{1,3}$, by the Witt decomposition, where $\dim_{\mathbb{C}} N = 2$. Pure spinors are defined by the property $x\mathbf{u} = 0$ for all $x \in N \subset \mathbb{C} \otimes \mathbb{R}^{1,3}$ [38]. In this context, Penrose flags can be defined by the expression $\text{Re}(i\mathbf{u}\tilde{\mathbf{u}})$ [22].

Now, the algebraic and formal properties of ELKO spinor fields, as defined in [1, 2, 7, 8], are briefly explored. An ELKO Ψ corresponding to a plane wave with momentum $p = (p^0, \mathbf{p})$ can be written, without loss of generality, as $\Psi(p) = \lambda(\mathbf{p})e^{-ip \cdot x}$ (or $\Psi(p) = \lambda(\mathbf{p})e^{ip \cdot x}$) where

$$\lambda(\mathbf{p}) = \begin{pmatrix} i\Theta\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad (4)$$

and the Wigner’s spin-1/2 time reversal operator Θ satisfies $\Theta\mathfrak{J}\Theta^{-1} = -\mathfrak{J}^*$, where \mathfrak{J} denotes the generators of rotations [1]. Hereon, as in [1], the Weyl representation of γ^μ is used, i.e.,

$$\gamma_0 = \gamma^0 = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}, \quad -\gamma_k = \gamma^k = \begin{pmatrix} \mathbb{O} & -\sigma_k \\ \sigma_k & \mathbb{O} \end{pmatrix}, \quad (5)$$

where

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

σ_i are the Pauli matrices. Also,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^{0123} = -i\gamma_{0123} = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix}.$$

ELKO spinor fields are eigenspinors of the charge conjugation operator C , i.e., $C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})$, for

$$C = \begin{pmatrix} \mathbb{O} & i\Theta \\ -i\Theta & \mathbb{O} \end{pmatrix} K. \quad (6)$$

The operator K is responsible for the \mathbb{C} -conjugation of Weyl spinor fields appearing on the right. The plus sign stands for *self-conjugate* spinors, $\lambda^S(\mathbf{p})$, while the minus yields *anti self-conjugate* spinors, $\lambda^A(\mathbf{p})$. Explicitly, the complete form of ELKO spinor fields can be found by solving the equation of helicity $(\sigma \cdot \hat{\mathbf{p}})\phi^\pm = \pm\phi^\pm$ in the rest frame and subsequently make a boost, to recover the result for any \mathbf{p} [1]. Here $\hat{\mathbf{p}} := \mathbf{p}/\|\mathbf{p}\|$. The four spinor fields are given

$$\lambda_{\{\mp, \pm\}}^{S/A}(p) = \sqrt{\frac{E+m}{2m}} \left(1 \mp \frac{\mathbf{p}}{E+m} \right) \lambda_{\{\mp, \pm\}}^{S/A}(\mathbf{0}), \quad (7)$$

where

$$\bar{\lambda}_{\{\mp, \pm\}}^{S/A}(\mathbf{p}) = \pm i \left[\lambda_{\{\pm, \mp\}}^{S/A}(\mathbf{p}) \right]^\dagger \gamma^0. \quad (8)$$

Note that, since $\Theta[\phi^\pm(\mathbf{0})]^*$ and $\phi^\pm(\mathbf{0})$ have opposite helicities, ELKO cannot be an eigenspinor field of the helicity operator, and indeed carries both helicities. In order to guarantee an invariant real norm, as well as positive definite norm for two ELKO spinor fields, and negative definite norm for the other two, the ELKO dual is given by

$$\lambda_{\{\mp, \pm\}}^{\neg S/A}(\mathbf{p}) = \pm i \left[\lambda_{\{\pm, \mp\}}^{S/A}(\mathbf{p}) \right]^\dagger \gamma^0. \quad (9)$$

Omitting the subindex of the spinor field $\phi_L(\mathbf{p})$, which is denoted hereon by ϕ , the left-handed spinor field $\phi_L(\mathbf{p})$ can be represented by

$$\phi = \begin{pmatrix} \alpha(\mathbf{p}) \\ \beta(\mathbf{p}) \end{pmatrix}, \quad \alpha(\mathbf{p}), \beta(\mathbf{p}) \in \mathbb{C}. \quad (10)$$

Now using Eqs.(2) it is possible to *calculate* explicitly the bilinear covariants for ELKO spinor fields¹¹:

$$\begin{aligned} \hat{\sigma} &= \lambda^\dagger \gamma_0 \lambda = 0, & \hat{\omega} &= -\lambda^\dagger \gamma_0 \gamma_{0123} \lambda = 0 \\ \hat{\mathbf{J}} &= \hat{J}_\mu \theta^\mu = \lambda^\dagger \gamma_0 \gamma_\mu \lambda \theta^\mu \neq 0 \\ \hat{\mathbf{K}} &= \hat{K}_\mu \theta^\mu = \lambda^\dagger i \gamma_{123} \gamma_\mu \lambda \theta^\mu = 0, \\ \hat{\mathbf{S}} &= \frac{1}{2} \hat{S}_{\mu\nu} \theta^{\mu\nu} = \frac{1}{2} \lambda^\dagger \gamma_0 i \gamma_{\mu\nu} \lambda \theta^{\mu\nu} \neq 0. \end{aligned}$$

Indeed, since the relations

$$\begin{aligned} \sigma_1 \phi &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, & \sigma_2 \phi &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix}, \\ \sigma_3 \phi &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}. \end{aligned}$$

¹¹ All the details are presented in [7].

hold, Eq.(4) gives $\psi^\dagger = [(\sigma_2 \phi^*)^\dagger, \phi^\dagger] = [(i\beta, -i\alpha), (\alpha^*, \beta^*)]$, and thus

$$\begin{aligned}\dot{\sigma} &= \psi^\dagger \gamma_0 \psi \\ &= [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} \alpha \\ \beta \\ -i\beta^* \\ i\alpha^* \end{pmatrix} \\ &= i\beta\alpha - i\alpha\beta - i\alpha^*\beta^* + i\beta^*\alpha^* \\ &= 0,\end{aligned}$$

$$\begin{aligned}\dot{\omega} &= -\psi^\dagger \gamma_{123} \psi = [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} i\alpha \\ i\beta \\ -\beta^* \\ \alpha^* \end{pmatrix} \\ &= 0,\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{J}} &= \dot{J}_\mu \gamma^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \gamma^\mu \\ &= \psi^\dagger \gamma_0 \gamma_1 \psi \gamma^1 + \psi^\dagger \gamma_0 \gamma_2 \psi \gamma^2 + \psi^\dagger \gamma_0 \gamma_3 \psi \gamma^3 - \psi^\dagger \psi \gamma^0 \\ &= \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \psi \gamma^1 + \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \psi \gamma^2 \\ &\quad + \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \psi \gamma^3 - \psi^\dagger \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \psi \gamma^0 \\ &= \psi^\dagger \begin{pmatrix} i\sigma_3 \phi^* \\ -\sigma_1 \phi \end{pmatrix} \gamma^1 + \psi^\dagger \begin{pmatrix} -\phi^* \\ -\sigma_2 \phi \end{pmatrix} \gamma^2 - \sigma^\dagger \begin{pmatrix} -i\sigma_1 \phi^* \\ -\sigma_3 \phi \end{pmatrix} + \psi^\dagger \psi \gamma^0 \\ &= [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} i\alpha^* \\ -i\beta^* \\ -\beta \\ -\alpha \end{pmatrix} \gamma^1 + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -\alpha^* \\ -\beta^* \\ i\beta \\ -i\alpha \end{pmatrix} \gamma^2 + \\ &\quad [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\beta^* \\ -i\alpha^* \\ -\beta \\ \alpha \end{pmatrix} \gamma^3 + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\beta^* \\ i\alpha^* \\ \alpha \\ \beta \end{pmatrix} \gamma^0 \\ &= 2(\alpha\beta^* + \alpha^*\beta)\gamma^1 + 2i(\alpha^*\beta - \alpha\beta^*)\gamma^2 + 2(\beta\beta^* - \alpha\alpha^*)\gamma^3 \\ &\quad + 2(\alpha\alpha^* + \beta\beta^*)\gamma^0 \\ &\neq 0,\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{K}} &= \dot{K}_\mu \gamma^\mu = \psi^\dagger i\gamma_{123} \gamma_\mu \psi \gamma^\mu \\ &= i\psi^\dagger \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} \psi \gamma^1 + i\psi^\dagger \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \psi \gamma^2 \\ &\quad + i\psi^\dagger \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix} \psi \gamma^3 + i\psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi \gamma^0 \\ &= \psi^\dagger \begin{pmatrix} i\sigma_3 \phi^* \\ \sigma_1 \phi \end{pmatrix} \gamma^1 - \psi^\dagger \begin{pmatrix} \phi^* \\ -\sigma_2 \phi \end{pmatrix} \gamma^2 - \psi^\dagger \begin{pmatrix} i\sigma_1 \phi^* \\ -\sigma_3 \phi \end{pmatrix} \gamma^3 + \psi^\dagger \begin{pmatrix} \sigma_2 \phi^* \\ -\phi \end{pmatrix} \gamma^0 \\ &= [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} \alpha^* \\ -\beta^* \\ -i\beta \\ -i\alpha \end{pmatrix} \gamma^1 + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} i\alpha^* \\ i\beta^* \\ \alpha \\ -\beta \end{pmatrix} \gamma^2 \\ &\quad + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -\beta^* \\ -\alpha^* \\ -i\alpha \\ i\beta \end{pmatrix} \gamma^3 + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\beta^* \\ i\alpha^* \\ \alpha \\ \beta \end{pmatrix} \gamma^0 \\ &= 0.\end{aligned}$$

Finally, the value for $\mathring{\mathbf{S}}$ is now computed:

$$\begin{aligned}
\mathring{\mathbf{S}} &= \frac{1}{2} \mathring{S}_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \gamma^{\mu\nu} \\
&= \frac{i}{2} (\psi^\dagger \gamma_1 \psi \gamma^{01} + \psi^\dagger \gamma_2 \psi \gamma^{02} + \psi^\dagger \gamma_3 \psi \gamma^{03} + \psi^\dagger \gamma_{012} \psi \gamma^{12} + \psi^\dagger \gamma_{013} \psi \gamma^{13} + \psi^\dagger \gamma_{023} \psi \gamma^{23}) \\
&= \frac{i}{2} \left(\psi^\dagger \begin{pmatrix} -\sigma_1 \phi^* \\ i\sigma_3 \phi \end{pmatrix} \gamma^{01} - \psi^\dagger \begin{pmatrix} \sigma_2 \phi \\ \phi^* \end{pmatrix} \gamma^{02} - \psi^\dagger \begin{pmatrix} \sigma_3 \phi \\ i\sigma_1 \phi^* \end{pmatrix} \gamma^{03} - \psi^\dagger \begin{pmatrix} i\sigma_3 \phi \\ \sigma_1 \phi^* \end{pmatrix} \gamma^{12} \right. \\
&\quad \left. + \frac{i}{2} \psi^\dagger \begin{pmatrix} -i\sigma_1 \phi \\ \sigma_3 \phi^* \end{pmatrix} \gamma^{23} - \frac{1}{2} \psi^\dagger \begin{pmatrix} \sigma_2 \phi \\ -\phi^* \end{pmatrix} \gamma^{13} \right) \\
&= \frac{i}{2} \{ [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -\beta^* \\ -\alpha^* \\ i\alpha \\ -i\beta \end{pmatrix} \gamma^{01} + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} i\beta \\ -i\alpha \\ -\alpha^* \\ -\beta^* \end{pmatrix} \gamma^{02} \\
&\quad + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -\alpha \\ \beta \\ -i\beta^* \\ -i\alpha^* \end{pmatrix} \gamma^{03} + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\alpha \\ i\beta \\ -\beta^* \\ -\alpha^* \end{pmatrix} \gamma^{12} \\
&\quad + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\beta \\ i\alpha \\ -\alpha^* \\ -\beta^* \end{pmatrix} \gamma^{13} + [(i\beta, -i\alpha), (\alpha^*, \beta^*)] \begin{pmatrix} -i\beta \\ -i\alpha \\ \alpha^* \\ -\beta^* \end{pmatrix} \gamma^{23} \} \\
&= \frac{i}{2} ((\alpha^*)^2 + (\beta^*)^2 - \beta^2 - \alpha^2) \gamma^{02} + \frac{1}{2} ((\alpha^*)^2 + (\beta^*)^2 + \beta^2 + \alpha^2) \gamma^{31} \\
&\quad + \frac{1}{2} ((\beta^*)^2 + \beta^2 - (\alpha^*)^2 - \alpha^2) \gamma^{01} + \frac{i}{2} (-\beta^2 - \alpha^2 + (\alpha^*)^2 + (\beta^*)^2) \gamma^{02} \\
&\quad + (\alpha\beta + \alpha^* \beta^*) \gamma^{03} + \frac{i}{2} (\alpha\beta - \alpha^* \beta^*) \gamma^{12} + \frac{i}{2} (\beta^2 - \alpha^2 + (\alpha^*)^2 - (\beta^*)^2) \gamma^{23}. \\
&\neq 0.
\end{aligned}$$

From these formulæ it is trivially seen that that $\mathring{\mathbf{J}}_\mu \mathring{\mathbf{K}} = 0$. The relations above give $\mathring{\mathbf{J}}^2 = 0$, and it is immediate that all Fierz identities introduced by the formulæ in Eqs.(3) are trivially satisfied.

It is useful to choose $i\Theta = \sigma_2$, as in [1], in such a way that it is possible to express

$$\lambda = \begin{pmatrix} \sigma_2 \phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}. \quad (11)$$

Now, any flagpole spinor field is an eigenspinor field of the charge conjugation operator [5, 6], which explicit action on a spinor ψ is given by $\mathcal{C}\psi = -\gamma^2 \psi^*$. Indeed using Eq.(11) it follows that

$$\begin{aligned}
-\gamma^2 \lambda^* &= \begin{pmatrix} \sigma_2 \phi^* \\ -\sigma_2 \sigma_2^* \phi \end{pmatrix} \\
&= \lambda.
\end{aligned}$$

Once the definition of ELKO spinor fields is recalled, we return to the previous discussion about Penrose flagpoles. Here we extend the definition of the Penrose poles, and we can prove that they are given in terms of an ELKO spinor field by the expression $\frac{1}{2} \langle \lambda(\widetilde{\gamma_{0123}} \lambda) \rangle_1$, and further, Penrose flags F can also be written in terms of ELKO, as $F = \frac{1}{2} \langle \lambda(\widetilde{\gamma_{0123}} \lambda) \rangle_2$. This assertion can be demonstrated following a reasoning analogous to the one exposed in [22, 39].

IV. CLASSICAL, IDEAL AND OPERATOR SPINORS IN THE SPACETIME ALGEBRA

Given an orthonormal basis $\{\mathbf{e}_\mu\}$ in $\mathbb{R}^{1,3}$ an arbitrary element of $\mathcal{Cl}_{1,3}$ is written as

$$\begin{aligned}
\Upsilon &= c + c^0 \mathbf{e}_0 + c^1 \mathbf{e}_1 + c^2 \mathbf{e}_2 + c^3 \mathbf{e}_3 + c^{01} \mathbf{e}_{01} + c^{02} \mathbf{e}_{02} + c^{03} \mathbf{e}_{03} + c^{12} \mathbf{e}_{12} + c^{13} \mathbf{e}_{13} \\
&\quad + c^{23} \mathbf{e}_{23} + c^{012} \mathbf{e}_{012} + c^{013} \mathbf{e}_{013} + c^{023} \mathbf{e}_{023} + c^{123} \mathbf{e}_{123} + c^{0123} \mathbf{e}_{0123}.
\end{aligned}$$

where $i\mathbf{e}_{0123} = \mathbf{e}_5$, and $\mathbf{e}_\mu \mathbf{e}_5 = -\mathbf{e}_5 \mathbf{e}_\mu$. From the isomorphism $\mathcal{Cl}_{1,3} \simeq \mathcal{M}(2, \mathbb{H})$, in order to obtain a representation of $\mathcal{Cl}_{1,3}$ the primitive idempotent $f = \frac{1}{2}(1 + \mathbf{e}_0)$ is used. The left minimal ideal of $\mathcal{Cl}_{1,3}$ is written as $I_{1,3} = \mathcal{Cl}_{1,3} f$, and an arbitrary element of $I_{1,3}$ is given by

$$I_{1,3} \ni \Xi = (a^1 + a^2 \mathbf{e}_{23} + a^3 \mathbf{e}_{31} + a^4 \mathbf{e}_{12}) f + (a^5 + a^6 \mathbf{e}_{23} + a^7 \mathbf{e}_{31} + a^8 \mathbf{e}_{12}) \mathbf{e}_5 f,$$

where [40]

$$\begin{aligned} a^1 &= c + c^0, & a^2 &= c^{23} + c^{023}, & a^3 &= -c^{13} - c^{013}, & a^4 &= c^{12} + c^{012}, \\ a^5 &= -c^{123} + c^{0123}, & a^6 &= c^1 - c^{01}, & a^7 &= c^2 - c^{02}, & a^8 &= c^3 - c^{03}. \end{aligned}$$

Denoting $\mathbf{i} = \mathbf{e}_{23}$, $\mathbf{j} = \mathbf{e}_{31}$, and $\mathbf{k} = \mathbf{e}_{12}$ it is immediate to see that the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfies the quaternion algebra — under the Clifford product — and

$$\mathcal{C}\ell_{1,3}f = I_{1,3} \ni \Xi = (a^1 + a^2\mathbf{i} + a^3\mathbf{j} + a^4\mathbf{k})f + (a^5 + a^6\mathbf{i} + a^7\mathbf{j} + a^8\mathbf{k})e_5f.$$

The set $\{1, e_5\}f$ is a basis for the ideal $I_{1,3}$, being possible to write $\mathbf{e}_\mu = f\mathbf{e}_\mu f + f\mathbf{e}_\mu e_5f - f\mathbf{e}_5\mathbf{e}_\mu f - f\mathbf{e}_5\mathbf{e}_\mu e_5f$. The matrix representation of the orthonormal basis e_μ is then obtained:

$$\mathbf{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix} \quad (12)$$

and the idempotents $f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{e}_5f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ can be obtained.

Using these representations, it is possible to write $\Upsilon \in \mathcal{C}\ell_{1,3}$ as

$$\begin{aligned} \Upsilon &= \begin{pmatrix} (c + c^0) + (c^{23} + c^{023})\mathbf{i} & (-c^{123} - c^{0123}) + (c^1 + c^{01})\mathbf{i} + \\ + (-c^{13} - c^{013})\mathbf{j} + (c^{12} + c^{012})\mathbf{k} & (c^2 + c^{02})\mathbf{j} + (c^3 + c^{03})\mathbf{k} \\ (-c^{123} + c^{0123}) + (c^1 - c^{01})\mathbf{i} & (c - c^0) + (c^{23} - c^{023})\mathbf{i} + \\ + (c^2 - c^{02})\mathbf{j} + (c^3 - c^{03})\mathbf{k} & (-c^{13} + c^{013})\mathbf{j} + (c^{12} - c^{012})\mathbf{k} \end{pmatrix} \\ &= \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}. \end{aligned} \quad (13)$$

In terms of the reversion in $\mathcal{C}\ell_{1,3}$ the matrix representation of Υ is given by

$$\tilde{\Upsilon} = \begin{pmatrix} \bar{q}_1 & -\bar{q}_3 \\ -\bar{q}_2 & \bar{q}_4 \end{pmatrix}, \quad (14)$$

where \bar{q} denotes the quaternionic conjugation.

The problem of representing spinor fields by completely skew-symmetric tensor fields (differential forms) comes back to Ivanenko, Landau and Fock in 1928, and was considered several times [5, 6, 18, 19, 30, 41]. An element $\Psi \in \mathcal{C}\ell_{1,3}^+$ — which corresponds to an operator spinor — can be written as

$$\mathcal{C}\ell_{1,3}^+ \ni \Psi = c + c^{01}e_{01} + c^{02}e_{02} + c^{03}e_{03} + c^{12}e_{12} + c^{13}e_{13} + c^{23}e_{23} + c^{0123}e_{0123}. \quad (15)$$

which in the light of the quaternionic representation in Eq.(13) is given by

$$\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} = \begin{pmatrix} c + c^{23}\mathbf{i} - c^{13}\mathbf{j} + c^{12}\mathbf{k} & c^{0123} - c^{01}\mathbf{i} - c^{02}\mathbf{j} - c^{03}\mathbf{k} \\ -c^{0123} + c^{01}\mathbf{i} + c^{02}\mathbf{j} + c^{03}\mathbf{k} & c + c^{23}\mathbf{i} + c^{13}\mathbf{j} + c^{12}\mathbf{k} \end{pmatrix}.$$

Now, considering the isomorphism $\mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3} \frac{1}{2}(1 + e_0) \simeq \mathbb{C}^4 \simeq \mathbb{H}^2$, it explicits the equivalence among the classical, the operatorial, and the algebraic definitions of a spinor [22, 23, 28, 38, 39, 41, 42, 43, 44, 45]. In this sense, the spinor space \mathbb{H}^2 carries the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $\text{SL}(2, \mathbb{C})$, it is isomorphic to the minimal left ideal $\mathcal{C}\ell_{1,3} \frac{1}{2}(1 + e_0)$ — corresponding to the algebraic spinor — and also isomorphic to the even subalgebra $\mathcal{C}\ell_{1,3}^+$ — corresponding to the operatorial spinor. It is then possible to write a Dirac spinor field as

$$\begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \frac{1}{2}(1 + \mathbf{e}_0) = \begin{pmatrix} c + c^{23}\mathbf{i} - c^{13}\mathbf{j} + c^{12}\mathbf{k} \\ c^{0123} - c^{01}\mathbf{i} - c^{02}\mathbf{j} - c^{03}\mathbf{k} \end{pmatrix} \in \mathbb{H} \oplus \mathbb{H}. \quad (16)$$

Returning to Eq.(15), and using for instance the standard representation Ψ can be represented by

$$\begin{pmatrix} c - ic^{12} & c^{13} - ic^{23} & -c^{03} + i^{0123} & -c^{01} + ic^{02} \\ -c^{13} - ic^{23} & c + ic^{12} & -c^{01} - ic^{02} & c^{03} + ic^{0123} \\ -c^{03} + ic^{0123} & -c^{01} + ic^{02} & c - ic^{12} & c^{13} - ic^{23} \\ -c^{01} - ic^{02} & c^{03} + ic^{0123} & -c^{13} - ic^{23} & c + ic^{12} \end{pmatrix} := \begin{pmatrix} \phi_1 & -\phi_2^* & \phi_3 & \phi_4^* \\ \phi_2 & \phi_1^* & \phi_4 & -\phi_3^* \\ \phi_3 & \phi_4^* & \phi_1 & -\phi_2^* \\ \phi_4 & -\phi_3^* & \phi_2 & \phi_1^* \end{pmatrix}.$$

The Dirac spinor ψ is an element of the minimal left ideal $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$, where¹² $f = \frac{1}{4}(1 + \gamma_0)(1 + i\gamma_{12})$. We choose take the Dirac standard representation that sends the basis vectors \mathbf{e}_μ to $\gamma_\mu \in \text{End}(\mathbb{C}^4)$. Then,

$$\psi = \Phi \frac{1}{2}(1 + i\gamma_{12}) \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f, \quad (17)$$

where $\Phi = \Phi \frac{1}{2}(1 + \gamma_0) \in \mathcal{C}\ell_{1,3}(1 + \gamma_0)$ is two times the real part of ψ . Using the matrix representation it follows that

$$(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f \ni \psi \simeq \mathbb{C} \otimes \begin{pmatrix} \phi_1 & 0 & 0 & 0 \\ \phi_2 & 0 & 0 & 0 \\ \phi_3 & 0 & 0 & 0 \\ \phi_4 & 0 & 0 & 0 \end{pmatrix} \simeq \mathbb{C} \otimes \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4, \quad (18)$$

where it can be seen the direct correspondence between ψ and the classical Dirac spinor.

V. MAPPING DIRAC TO ELKO SPINOR FIELDS AND THE INSTANTON HOPF FIBRATION

The suitable mathematical structure to describe the instanton is a principal bundle with base manifold S^4 and associated structural group $\text{SU}(2)$. In [18] a formalism similar to the magnetic monopole was exhibited and constructed, exploring the relationship between spinor fields and the bilinear covariants. Spinor fields indirectly describe fermionic fields, since the observables are their associated bilinear covariants. In [46] a tomographic scheme — based on spacetime symmetries — was presented for the reconstruction of the internal degrees of freedom of a Dirac spinor, together with the possibility of the tomographic group be taken as $\text{SU}(2)$. In addition, in [32] the spinor field was reconstructed from the bilinear covariants.

As argued in [18], using the inversion theorem for Euclidean signature, it is possible to formulate those constructions for the case of magnetic monopoles and instantons, indicating the generalizations of the Balachandran's construction to the case of instantons [20]. Also, the inversion theorem for Minkowski spacetime appeared for the first time in the paper [21]. On the other hand, in a previous paper [17] we investigate and provide the necessary and sufficient conditions to map Dirac spinor fields (DSFs) to ELKO, in order to naturally extend the Standard Model to spinor fields possessing mass dimension one. Let us make a briefly review of which are the conditions a Dirac spinor field must obey to be led to an ELKO. In [17] there has been proved that not all DSFs can be led to ELKO, but only a subset of the three classes — under Lounesto classification — of DSFs restricted to some conditions. Explicitly, by taking a DSF

$$\psi(\mathbf{p}) = \begin{pmatrix} \phi_R(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \epsilon \sigma_2 \phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad (19)$$

and taking into account that $\phi_R(\mathbf{p}) = \chi \phi_L(\mathbf{p})$, where $\chi = \frac{E + \sigma \cdot \mathbf{p}}{m}$ and $\kappa \psi = \psi^*$, and denoting the 4-component DSF by $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ ($\psi_r \in \mathbb{C}, r = 1, \dots, 4$), we have the simultaneous conditions a DSF must obey in order for it to be led to an ELKO [17]:

$$\begin{aligned} 0 &= \text{Re}(\psi_1^* \psi_3) = \text{Re}(\psi_2^* \psi_4) \\ 0 &= \text{Re}(\psi_2^* \psi_3) + \text{Re}(\psi_1^* \psi_4) \\ 0 &= \text{Im}(\psi_1^* \psi_4) - \text{Im}(\psi_2^* \psi_3) - 2\text{Im}(\psi_3^* \psi_4) - 2\text{Im}(\psi_1^* \psi_2). \end{aligned} \quad (20)$$

In what follows we obtain the extra necessary and sufficient conditions for each class of DSFs.

As additional conditions on class-(2) Dirac spinors, we also have:

$$\text{Re}(\psi_1^* \psi_4) + \text{Im}(\psi_2^* \psi_3) = 0. \quad (21)$$

For the class-(3) of spinor fields, the additional condition was obtained in [17]:

$$\text{Im}(\psi_1^* \psi_4) - \text{Im}(\psi_2^* \psi_3) - 2\text{Im}(\psi_1^* \psi_2) = 0. \quad (22)$$

¹² We choose to express $f = \frac{1}{4}(1 + \gamma_0)(1 + i\gamma_{12})$ using the Dirac representation. It could be chosen the idempotent $f = \frac{1}{4}(1 + \gamma_5)(1 + i\gamma_{12})$ associated with the Weyl representation.

Class-(1) DSFs must obey all the conditions given by Eqs.(20), (21), and (22). Note that if one relaxes the condition given by Eq.(21) or Eq.(22), DSFs of types-(3) and -(2) are respectively obtained.

Using the decomposition $\psi_j = \psi_{ja} + i\psi_{jb}$ (where $\psi_{ja} = \text{Re}(\psi_j)$ and $\psi_{jb} = \text{Im}(\psi_j)$) it follows that $\text{Re}(\psi_i^* \psi_j) = \psi_{ia}\psi_{ja} + \psi_{ib}\psi_{jb}$ and $\text{Im}(\psi_i^* \psi_j) = \psi_{ia}\psi_{jb} - \psi_{ib}\psi_{ja}$ for $i, j = 1, \dots, 4$. So, in components, the conditions in common for all types of DSFs are

$$\psi_{1a}\psi_{3a} + \psi_{1b}\psi_{3b} = 0, \quad (23)$$

$$\psi_{2a}\psi_{4a} + \psi_{2b}\psi_{4b} = 0, \quad (24)$$

and the additional conditions for each case are summarized in Table I below.

Class	Additional conditions
(1)	$\psi_{2a}(\psi_{3a} - \psi_{3b}) + \psi_{2b}(\psi_{3a} + \psi_{3b}) = 0 = \psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a}$
(2)	$\psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a} = 0 = \psi_{2a}\psi_{3a} + \psi_{2b}\psi_{3b} + \psi_{1a}\psi_{4a} + \psi_{1b}\psi_{4b}$
(3)	$\psi_{2a}(\psi_{3a} - \psi_{3b}) + \psi_{2b}(\psi_{3a} + \psi_{3b}) = 0$ and $(\psi_{1a}\psi_{4b} - \psi_{1b}\psi_{4a}) - (\psi_{2a}\psi_{3b} - \psi_{2b}\psi_{3a})$ $-2(\psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a}) - 2(\psi_{1a}\psi_{2b} - \psi_{1b}\psi_{2a}) = 0$

TABLE I: Additional conditions, in components, for class (1), (2) and (3) Dirac spinor fields.

The explicit mappings obtained above present the same form of the instanton Hopf fibration map $S^3 \dots S^7 \rightarrow S^4$ mapping obtained in [18], and could be interpreted as the geometric meaning of the mass dimension-transmuting operator obtained in [17], where we obtained mapping between ELKO and Dirac spinor fields. As the latter possess mass dimension 3/2, the former presents mass dimension 1. Some results involving the instanton Hopf fibration can also be seen in this context, e.g, in [47]. It could explain why ELKO spinor fields satisfy a Klein-Gordon equation, instead of the Dirac equation [1, 2, 3, 8, 9, 48].

Indeed, the monopole construction was based [18] on the Hopf fibration $S^1 \dots S^3 \rightarrow S^2$, where S^1 is homeomorphic to the Lie gauge group $U(1)$ of the electromagnetism. Using a similar construction [18], the instanton is related to a principal bundle with structure Lie group $SU(2)$, which is homeomorphic to the 3-sphere S^3 . The instanton was described in [18] using the the Hopf fibration $S^3 \dots S^7 \rightarrow S^4$ by means of the bilinear covariants associated with the Dirac spinor fields, under Lounesto spinor field classification.

Types-(1) and -(2) Dirac spinor fields can be regarded as satisfying $\sigma = 1$, which is exactly S^7 , when the Dirac spinor field is classically described by an element of $\mathbb{C}^4 \simeq \mathbb{H}^2$ (here these spaces are isomorphic as vector spaces). Now, the Fierz identities described in Eq.(3) give immediately — from the equation $\sigma = 1$ — the expression $\mathbf{J}^2 + \omega^2 = 1$, which is S^4 .

The mapping in Eq.(16) induces the possibility to interpret the coordinate 8-tuple $(c, c^{23}, -c^{13}, c^{12}, c^{0123}, -c^{01}, -c^{02}, -c^{03})$ as local coordinates in S^7 , and then S^7 is the (compact) space described by an unitary Dirac spinor. The Fierz identities imply that $\mathbf{J}^2 + \omega^2 = 1$, which is topologically an S^4 with local coordinates $(J_0, J_1, J_2, J_3, \omega)$.

Using the definition of the bilinear covariants in Eq.(2) and the quaternionic representation of the Dirac spinor in Eq.(16), it is possible to write [18]

$$\begin{aligned} \sigma &= \|q_1\|^2 + \|q_2\|^2, & \omega &= 2 \text{Re}(q_1^* q_2), & J_0 &= \|q_1\|^2 - \|q_2\|^2 \\ J_1 &= 2 \text{Re}(q_1^* i q_2), & J_2 &= 2 \text{Re}(q_1^* j q_2), & J_3 &= 2 \text{Re}(q_1^* k q_2), \end{aligned}$$

which in the representation given by Eq.(18) is given by [18]

$$\begin{aligned} \sigma &= \|\psi_1\|^2 + \|\psi_2\|^2 + \|\psi_3\|^2 + \|\psi_4\|^2 = 1 \\ J_0 &= \|\psi_1\|^2 + \|\psi_2\|^2 - \|\psi_3\|^2 - \|\psi_4\|^2 = 1 \\ J_1 &= 2\text{Im}(\psi_1\psi_4^*) + 2\text{Im}(\psi_2\psi_3^*) \\ J_2 &= 2\text{Re}(\psi_2\psi_3^*) - 2\text{Re}(\psi_1\psi_4^*) \\ J_3 &= 2\text{Im}(\psi_3\psi_1^*) + 2\text{Im}(\psi_2\psi_4^*) \\ \omega &= 2\text{Re}(\psi_1\psi_3^*) + 2\text{Re}(\psi_2\psi_4^*). \end{aligned} \quad (25)$$

Although these expressions are not the same as Eqs.(20, 21, 22) we might argue whether there is a corresponding application $M \in \text{End}(\mathbb{C}^4)$ leading Dirac to ELKO spinor fields that indeed corresponds to the expressions above, in the light of the procedure in [17]. In the paper [17] there is an explicit algorithm that constructs such an application, using straightforward assumptions. Even if we could keep the same application $M \in \text{End}(\mathbb{C}^4)$ as obtained in [17] and change the form of the Dirac spinor field, or take the same spinor field in Eq.(18) and construct another application $M' \in \text{End}(\mathbb{C}^4)$ — using an analogous procedure as explicitly exhibited in [17] — in such a way that the instanton Hopf fibration conditions in Eqs.(25) and the Dirac to ELKO mapping in Eqs.(20, 21, 22) be similar, we remember that in Eqs.(25) the terms J_0, J_1, J_2, J_3 cannot simultaneously equal zero, because $\mathbf{J} \neq 0$. Formally, the instanton cannot be described by an ELKO spinor field. This statement mathematically explain the well known physical interpretation that while the instanton is a localized topological object, ELKO is a non-local extended one [1].

VI. TYPE-(4) (FLAG-DIPOLE) SPINOR FIELDS

It has been argued that the flag-dipole spinor fields (type-(4) under Lounesto spinor field classification) are related to the quark confinement, although they are not appropriate to describe fermions, since they do not constitute a real vector space [5]. As ELKO spinor fields are prime candidates to describe dark matter, and the flag-dipole spinor fields can shed some new light on the quark confinement investigations, we want to point out some algebraic and geometric considerations concerning the type-(4) spinor fields.

The Weyl and Majorana spinor fields can be written in terms of operator spinor fields as

$$\Psi \frac{1}{2}(1 + \gamma_0 \mathbf{u}), \quad \Psi \in \sec \mathcal{C}^+ \ell(M, \eta),$$

where $\mathcal{C}^+ \ell(M, \eta)$ denotes the spacetime Clifford bundle, in which the typical fiber is $\mathcal{C}^+ \ell_{1,3}$ where $\mathbf{u} = \pm \gamma_2$ for Weyl spinor fields and $\mathbf{u} = \pm \gamma_1$ for Majorana spinor fields.

More generally, ELKO spinor fields can also be written in the same form, as

$$\Psi \frac{1}{2}(1 + \gamma_0 \mathbf{u}), \quad \Psi \in \sec \mathcal{C}^+ \ell(M, \eta)$$

where \mathbf{u} propitiates a mixture of Weyl and Majorana spinor fields ($\mathbf{u} = \gamma_1 \cos \alpha + \gamma_2 \sin \alpha$). This mixture can be written as $\mathbf{u} = \gamma_1 \cos \phi + \mathbf{i} \gamma_3 \sin \phi$, where $\mathbf{i} = -\gamma_2 \gamma_3$.

In addition, following Doran's conjecture [5], all the flag-dipole — type-(4) spinor fields under Lounesto spinor field classification — can be written in a similar form as

$$\Psi \frac{1}{2}(1 + \gamma_0 \mathbf{u}), \quad \Psi \in \mathcal{C}^+ \ell(M, \eta), \quad \mathbf{u} \in \mathbb{R}^3, \quad \mathbf{u}^2 = -1.$$

More precisely, it is assumed that \mathbf{u} is a spatial unit vector ($\mathbf{u} \cdot \gamma_0 = -1$), and \mathbf{u} is neither a multiple of γ_3 nor a multiple of $\gamma_1 \gamma_2$.

Now, by introducing the complex multivector field as in [5, 6] $Z \in \sec \mathbb{C} \ell(M, \eta)$ (where $\mathbb{C} \ell(M, \eta)$ denotes the complexified spacetime Clifford bundle, in which the typical fiber is $\mathbb{C} \otimes \mathcal{C} \ell_{1,3} \simeq \mathcal{C} \ell_{4,1}$ [23]) and the corresponding complex multivector *operator* (represented by the same letter):

$$Z = \sigma + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \omega\gamma_{0123}. \quad (26)$$

When the multivector operators $\sigma, \omega, \mathbf{J}, \mathbf{S}, \mathbf{K}$ satisfy the Fierz identities, then the complex multivector operator Z is denominated a *Fierz aggregate*, and, when $\gamma_0 Z^\dagger \gamma_0 = Z$, which means that Z is a Dirac self-adjoint aggregate¹³, Z is called a *boomerang*.

A spinor field such that *not both* ω and σ are null is said to be regular. When $\omega = 0 = \sigma$, a spinor field is said to be *singular*. In this case the Fierz identities are in general replaced by the more general conditions [32] (which obviously also hold for $\omega, \sigma \neq 0$). These conditions are:

$$\begin{aligned} Z^2 &= 4\sigma Z, & Z\gamma_\mu Z &= 4J_\mu Z, & Zi\gamma_{\mu\nu} Z &= 4S_{\mu\nu} Z, \\ Zi\gamma_{0123}\gamma_\mu Z &= 4K_\mu Z, & Z\gamma_{0123} Z &= -4\omega Z. \end{aligned} \quad (27)$$

¹³ It is equivalent to say that $\omega, \sigma, \mathbf{J}, \mathbf{K}, \mathbf{S}$ are real multivector fields.

Now, any spinor field (regular or singular) can be reconstructed from its bilinear covariants as follows. Take an arbitrary spinor field ξ satisfying $\xi^\dagger \gamma_0 \psi \neq 0$. Then the spinor field ψ and the multivector field $Z\xi$, differ only by a phase. Indeed, it can be written as

$$\psi = \frac{1}{4N} e^{-i\alpha} Z\xi,$$

where $N = \frac{1}{2} \sqrt{\xi^\dagger \gamma_0 Z\xi}$ and $e^{-i\alpha} = \frac{1}{N} \xi^\dagger \gamma_0 \psi$. For more details see, e.g., [32, 46].

For the specific case of type-(4) (flag-dipole) spinor fields, the boomerang can be written as

$$Z = \mathbf{J} + i\mathbf{J}s - ih\gamma_{0123}\mathbf{J}, \quad \mathbf{J}^2 = 0 \quad (28)$$

where s denotes a spacelike vector orthogonal to \mathbf{J} , meaning that $s^2 < 0$ and $\mathbf{J} \lrcorner s = 0$. The bilinear covariant \mathbf{S} is given by the Clifford product $\mathbf{S} = \mathbf{J}s = \mathbf{J} \wedge s$. Exclusively for the type-(4) flag-dipole spinor fields, the real coefficient satisfies $h \neq 0$. For all other types of spinor fields, including type-(4), h is constrained to s by the expression

$$h^2 = 1 + s^2 < 1, \quad \text{since } s^2 < 0 \quad (29)$$

and is defined as to relate the two bilinear covariants \mathbf{K} and \mathbf{J} by $\mathbf{K} = h\mathbf{J}$.

Using Eq.(28), it is immediate to verify that $(1 + is - ih\gamma_{0123})Z = 0$ and also that the boomerang Z is a lightlike Clifford multivector, i.e., $Z^2 = 0$, since flag-dipole spinor fields in class-(4) under Lounesto spinor field classification satisfies $\mathbf{J}^2 = 0$ (see Eq.(2)). Doran's conjecture asserts that the coefficient h is given by $\mathbf{u} \cdot \gamma_3$ [5, 6]. Also, the equation $Z^2 = 0$ implies that $Z = \mathbf{J}(1 + is + ih\gamma_{0123})$.

Furthermore, using the representation of the type-(4) flag-dipole spinor field ψ as an element of the minimal left ideal $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})\frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_1\gamma_2)$ it follows that [5, 6] $\frac{1}{2}(1 - is - ih\gamma_{0123})\Psi = \Psi$, while $\frac{1}{2}(1 + is + ih\gamma_{0123})\Psi = 0$.

Now, by means of the isomorphism $\mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + \gamma_0) \simeq \mathbb{C}^4$ Lounesto defined a projector $\Sigma_\pm \in \text{End}(\mathcal{C}\ell_{1,3})$ by the expression

$$\Sigma_\pm(u) = \frac{1}{2}(u \pm (s \mp \gamma_{0123}\mathbf{J}\mathbf{K}^{-1}u\gamma_{0123})), \quad (30)$$

in such a way that this definition keep unaltered an ideal (algebraic) spinor in the minimal left ideal $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$. As for ideal spinor fields ψ the equation $\psi\gamma_{0123} = \psi\gamma_2\gamma_1$, for the case where ψ is a type-(4) flag-dipole spinor Ψ , the relation $\frac{1}{2}(1 + is + ih\gamma_{0123})\Psi = 0$ holds, as we have just seen. In this case, $\mathbf{K} = h\mathbf{J}$, and the projector $\Sigma_\pm \in \text{End}(\mathcal{C}\ell_{1,3})$ acts on Ψ as

$$\Sigma_\pm(\Psi) = \frac{1}{2}(\Psi \pm (s + h\gamma_{0123})\Psi\gamma_1\gamma_2). \quad (31)$$

ELKO spinors $\lambda(\mathbf{p})$ are obtained as a particular case where $h = 0$. Indeed, as type-(4) spinor fields are defined by the relations assumed by their bilinear covariants $\omega = 0 = \sigma$, $\mathbf{K} \neq 0$, $\mathbf{J} \neq 0$. As $\mathbf{K} = h\mathbf{J}$, when we put $h = 0$, the bilinear covariants assume the expressions $\omega = 0 = \sigma$, $\mathbf{K} = 0$, $\mathbf{J} \neq 0$, which are precisely the bilinear covariants associated with ELKO (type-(5)) spinor fields. Then, ELKO spinor fields can be thought as being limiting cases of type-(4) spinor fields:

$$\lambda(\mathbf{p}) = \Sigma_\pm(\Psi) = \frac{1}{2}(\Psi \pm s\Psi\gamma_1\gamma_2). \quad (32)$$

Clearly, all the six classes under Lounesto spinor field classification are disjoint classes, and in particular, type-(4) flag-dipole spinor fields and type-(5) flagpole (ELKO) spinor fields are disjoint. The limit $h \rightarrow 0$ changes the class (4) into class (5).

Also, using Eq.(29), it is also possible to turn type-(4) flag-dipole spinor fields into type-(6) Weyl spinor fields, in the limit $s \rightarrow 0$ — implying that $h = \pm 1$, and Weyl spinor fields can be alternatively written as [5, 6]

$$\Sigma_\pm(\Psi) = \frac{1}{2}(\Psi \pm \gamma_{0123}\Psi\gamma_1\gamma_2). \quad (33)$$

In this sense, endomorphisms of $\mathcal{C}\ell_{1,3}$ where exhibited in [5, 6], which make possible to change spinor fields classes under Lounesto spinor field classification. Weyl and ELKO spinor fields are obtained from an arbitrary type-(4) flag-dipole spinor field, respectively corresponding to type-(6) and type-(5) spinor fields.

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Appendix A: Operator Spinors

Recall that given the \mathbb{Z}_2 -graded Clifford algebra $\mathcal{C}\ell_{1,3}$, we can use the even subalgebra $\mathcal{C}\ell_{1,3}^+ \simeq \mathcal{C}\ell_{3,0}$ as the representation space for $\mathcal{C}\ell_{p,q}$. Define a representation $\rho : \mathcal{C}\ell_{1,3} \rightarrow \text{End}(\mathcal{C}\ell_{1,3}^+)$, the so-called irreducible graded representation (IGR).

A multivector $\psi \in \mathcal{C}\ell_{1,3}$ can be split as $\psi = \psi_+ + \psi_-$, where $\psi_{\pm} = \frac{1}{2}(\psi \pm \hat{\psi}) \in \mathcal{C}\ell_{1,3}^{\pm}$. Consider now $\rho = \rho_+ + \rho_-$ and $\rho(\psi) = \rho_+(\psi_+) + \rho_-(\psi_-)$. For $\psi_- \in \mathcal{C}\ell_{p,q}^-$ it follows that $a_- \phi \in \mathcal{C}\ell_{1,3}^-$ for $\phi \in \mathcal{C}\ell_{1,3}^+$, i.e., $\rho_+(\psi_+)(\phi) = \psi_+ \phi$, $\forall \phi \in \mathcal{C}\ell_{1,3}^+$. Now take an odd element $\varsigma \in \mathcal{C}\ell_{1,3}^-$ and define $\rho_-(\psi_-)(\phi) = \psi_- \phi \varsigma$, $\forall \phi \in \mathcal{C}\ell_{1,3}^+$. If ς is chosen in such a way that $\varsigma^2 = 1$, where $\varsigma \in \mathcal{C}\ell_{1,3}^-$, the definition of IGR does depend on the existence of an odd element such that $\varsigma^2 = 1$. In the particular cases $\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$ and $\mathcal{C}\ell_{0,2} \simeq \mathbb{H}$, such an element does not exist. In order to show that ρ is irreducible, suppose that there exists an element $\varpi_1 \in \mathcal{C}\ell_{p,q}^+$ such that $(\varsigma_1)^2 = 1$ and $\varpi \varsigma = \varsigma \varpi$. We write $\mathcal{C}\ell_{1,3}^+ = +\mathcal{C}\ell_{1,3}^+ \oplus -\mathcal{C}\ell_{1,3}^+$, where $\pm\mathcal{C}\ell_{1,3}^+ = \mathcal{C}\ell_{1,3}^+ \frac{1}{2}(1 \pm \varpi_1)$, and, for $\phi_{\pm} \in \pm\mathcal{C}\ell_{1,3}^+$, it follows that $\phi_{\pm} \varpi_1 = \pm \phi_{\pm}$. Each one of the spaces $\pm\mathcal{C}\ell_{1,3}^+$ is invariant under ρ , as can be immediately seen by the relations $(\varsigma_1)^2 = 1$ and $\varpi \varsigma = \varsigma \varpi$, and in addition these subspaces are subalgebras of $\mathcal{C}\ell_{1,3}^+$.

If there exists another even element ϖ_2 such that $(\varpi_2)^2 = 1$, then $\varpi_2 \varpi_1 = \varpi_1 \varpi_2$ and $\varpi_2 \varsigma = \varsigma \varpi_2$. It follows that the subspaces $\pm\mathcal{C}\ell_{1,3}^+$ do not carry an irreducible representation. It is defined

$$\pm\pm\mathcal{C}\ell_{1,3}^+ = \pm\mathcal{C}\ell_{1,3}^+ \frac{1}{2}(1 \pm \varpi_1) \frac{1}{2}(1 \pm \varpi_2), \quad (34)$$

each one is invariant under ρ , i.e., $\rho(\pm\pm\mathcal{C}\ell_{1,3}^+) \hookrightarrow \pm\pm\mathcal{C}\ell_{p,q}^+$. It is possible to continue this construction in $\mathcal{C}\ell_{p,q}$ when there is another even element ϖ_3 such that $(\varpi_3)^2 = 1$, $\varpi_3 \varpi_1 = \varpi_1 \varpi_3$, $\varpi_3 \varpi_2 = \varpi_2 \varpi_3$ and $\varpi_3 \varsigma = \varsigma \varpi_3$.

When there is not even elements satisfying these conditions anymore, an irreducible representation is obtained. The space that carries such representations is called spinor algebra, a subalgebra of the even subalgebra. In some cases it can be the even subalgebra itself. An element of the IGR of $\mathcal{C}\ell_{1,3}$ is called an operator spinor.

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